**Cambridge Assessment International Education**
Cambridge International Advanced Level

**FURTHER MATHEMATICS** **9231/11**

Paper 1 **October/November 2017**

MARK SCHEME

Maximum Mark: 100

**Published**

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**Mark Scheme Notes**

Marks are of the following three types:

M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

B Mark for a correct result or statement independent of method marks.

- When a part of a question has two or more "method" steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly when there are several B marks allocated. The notation DM or DB (or dep*) is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

- The symbol FT implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only. A and B marks are not given for fortuitously "correct" answers or results obtained from incorrect working.

  - Note: B2 or A2 means that the candidate can earn 2 or 0.
    B2/1/0 means that the candidate can earn anything from 0 to 2.

  The marks indicated in the scheme may not be subdivided. If there is genuine doubt whether a candidate has earned a mark, allow the candidate the benefit of the doubt. Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored.

- Wrong or missing units in an answer should not lead to the loss of a mark unless the scheme specifically indicates otherwise.

- For a numerical answer, allow the A or B mark if a value is obtained which is correct to 3 s.f., or which would be correct to 3 s.f. if rounded (1 d.p. in the case of an angle). As stated above, an A or B mark is not given if a correct numerical answer arises fortuitously from incorrect working. For Mechanics questions, allow A or B marks for correct answers which arise from taking *g* equal to 9.8 or 9.81 instead of 10.

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The following abbreviations may be used in a mark scheme or used on the scripts:

| | |
|---|---|
| AEF/OE | Any Equivalent Form (of answer is equally acceptable) / Or Equivalent |
| AG | Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid) |
| CAO | Correct Answer Only (emphasising that no “follow through” from a previous error is allowed) |
| CWO | Correct Working Only – often written by a ‘fortuitous’ answer |
| ISW | Ignore Subsequent Working |
| SOI | Seen or implied |
| SR | Special Ruling (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance) |

**Penalties**

MR –1 A penalty of MR –1 is deducted from A or B marks when the data of a question or part question are genuinely misread and the object and difficulty of the question remain unaltered. In this case all A and B marks then become “follow through” marks. MR is not applied when the candidate misreads his own figures – this is regarded as an error in accuracy. An MR –2 penalty may be applied in particular cases if agreed at the coordination meeting.

PA –1 This is deducted from A or B marks in the case of premature approximation. The PA –1 penalty is usually discussed at the meeting.

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| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 1 | $\sum_{r=1}^{n} u_r = 16\sum_{r=1}^{n} r^2 - 8\sum_{r=1}^{n} r - 3n$ | **M1A1** | **M1** for split into 3 parts |
| | $= 16\frac{n(n+1)(2n+1)}{6} - 8\frac{n(n+1)}{2} - 3n$ | **M1** | For using formulae correctly in their expression |
| | $= \ldots = \frac{n}{3}\left(16n^2 + 12n - 13\right)$ (3 terms) | **A1** | OE |
| | | **4** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 2 | CF: $m^2 + 2m + 5 = 0 \Rightarrow m = -1 \pm 2\text{i}$ | **M1** | |
| | $e^{-t}(A \cos 2t + B \sin 2t)$ | **A1** | |
| | PI: $x = pt^2 + qt + r \Rightarrow \dot{x} = 2pt + q \Rightarrow \ddot{x} = 2p$ | **M1** | |
| | $2p + 4pt + 2q + 5pt^2 + 5qt + 5r = 4 - 5t^2$ | **M1** | |
| | $\Rightarrow p = -1,\ q = \frac{4}{5},\ r = \frac{22}{25}$ | **A1** | |
| | GS: $x = e^{-t}(A \cos 2t + B \sin 2t) + \frac{22}{25} + \frac{4}{5}t - t^2$ | **A1FT** | |
| | | **6** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 3(i) | $\frac{d^{n+1}}{dx^{n+1}}\left(x^{n+1}\ln x\right)=\frac{d^n}{dx^n}\left(x^{n+1}.\frac{1}{x}+(n+1)x^n\ln x\right)=$ $\frac{d^n}{dx^n}\left(x^n+(n+1)x^n\ln x\right)$ | **M1A1** | AG |
| | | **2** | |
| 3(ii) | Assume $H_k$ is true $\Rightarrow \frac{d^k}{dx^k}\left(x^k\ln x\right)=k!\left\{\ln x+1+\frac{1}{2}+\ldots+\frac{1}{k}\right\}$ | **B1** | Statement of $H_k$ seen |
| | $\frac{d^{k+1}}{dx^{k+1}}\left(x^{k+1}\ln x\right)=\frac{d^k}{dx^k}\left(x^k+[k+1]x^k\ln x\right)$ | **M1** | |
| | $=k!+[k+1]k!\left\{\ln x+1+\frac{1}{2}+\ldots+\frac{1}{k}\right\}$ | **A1** | |
| | $=(k+1)!\left\{\ln x+1+\frac{1}{2}+\ldots+\frac{1}{k+1}\right\}\Rightarrow H_{k+1}$ is true | **A1** | |
| | Check $H_1$ is true **and** $H_k$ is true $\Rightarrow H_{k+1}$ is true; hence, by PMI, $H_n$ is true for all positive integers $n$. | **A1** | |
| | | **5** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 4(i) | $\alpha+\beta+\gamma=\frac{3}{2}\quad \alpha\beta+\beta\gamma+\gamma\alpha=2\quad \alpha\beta\gamma=5+\beta+\gamma=$ $\frac{3}{2}\alpha\beta+\beta\gamma+\gamma\alpha=2\alpha\beta\gamma=5$ | **B1** | (Can be awarded in (ii) if not seen here)<br>SOI |
| | $(\alpha+1)(\beta+1)(\gamma+1)=\alpha\beta\gamma+(\alpha\beta+\beta\gamma+\gamma\alpha)+$ $(\alpha+\beta+\gamma)+1$ | **M1A1** | Multiply out **and** group for **M1** |
| | $=5+2+1\frac{1}{2}+1=9\frac{1}{2}$ | **A1FT** | Alt method:<br>Let $x=y-1$ **M1**<br>Sub and expand<br>$2y^3-9y^2\ 16y-19=0$ **M1**, **A1**<br>Product of roots = 19/2 **A1** |
| | | **4** | |
| 4(ii) | $(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta)=\left(1\frac{1}{2}-\alpha\right)\left(1\frac{1}{2}-\beta\right)\left(1\frac{1}{2}-\gamma\right)$ | **M1** | Alt methods: $=(\sum\alpha)(\sum\alpha\beta)-\alpha\beta\gamma$ or $\sum\alpha^2\sum\alpha+2\alpha\beta\gamma-\sum\alpha^3$ |
| | $=\frac{27}{8}-\frac{9}{4}(\alpha+\beta+\gamma)+\frac{3}{2}(\alpha\beta+\beta\gamma+\gamma\alpha)-\alpha\beta\gamma$ | **A1** | |
| | $=\frac{27}{8}-\frac{9}{4}\times\frac{3}{2}+\frac{3}{2}\times2-5=-2$ | **M1A1** | |
| | | **4** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 5(i) | $6x^2+6xy+3x^2y'-9y^2y'=0\ (*) \Rightarrow 2x(x+y)=(3y^2-x^2)y'$ | **M1A1** | |
| | $y'=0$ and $x\neq 0 \Rightarrow x=-y$ | **M1A1** | |
| | $\Rightarrow 2x^3-3x^3+3x^3=16 \Rightarrow A$ is $(2, -2)$ | **A1** | |
| | | **5** | |
| 5(ii) | $12x+6xy'+6y+6xy'+3x^2y''-\left[18y\left(y'\right)^2+9y^2y''\right]=0$ | ***M1** | |
| | $x=2 \quad y=-2 \quad y'=0 \Rightarrow 8-4+4y''-12y''=0$ | **DM1** | |
| | $\Rightarrow y''=\frac{1}{2}$ | **A1** | |
| | | **3** | |

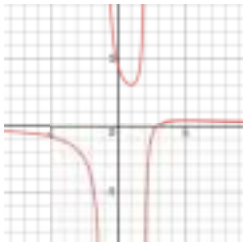
| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 6(i) | $\overrightarrow{AB} = \mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$ $\overrightarrow{BC} = -4\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$ $\overrightarrow{AC} = -3\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$ | **B1** | 2 correct required |
| | $\overrightarrow{AB} \times \overrightarrow{BC} = 21\mathbf{i} + 3\mathbf{j} + 18\mathbf{k}$ (*) | **M1A1** | OE |
| | Area of triangle $ABC = \frac{1}{2}\sqrt{21^2 + 3^2 + 18^2} = 13.9\left(\frac{3}{2}\sqrt{86}\right)$ | **A1** | |
| | Alt method**:** Use scalar product to find angle | **(M1A1** | |
| | Find area using Area = ½ $ab$ sin $C$ or equivalent | **M1A1)** | |
| | | **4** | |
| 6(ii) | $\mathrm{d} = \frac{\left\lvert \overrightarrow{AB} \times \overrightarrow{BC} \right\rvert}{\left\lvert \overrightarrow{BC} \right\rvert} = \frac{\sqrt{21^2 + 3^2 + 18^2}}{\sqrt{4^2 + 2^2 + 5^2}}$ | **M1A1** | Alt method: Find angle at $C$ |
| | $= 4.15\left(\frac{1}{5}\sqrt{430}\right)$ | **A1** | Area triangle = sin $C \times \lvert AC \rvert$ |
| | Alt method: Use equation of BC to find D (foot of perpendicular) in terms of parameter **and** scalar product to find parameter , λ= 8/15. Find length | **(M1A1)** | |
| | | **3** | |
| 6(iii) | From (*) Cartesian equation is $7x + y + 6z = \text{const.}$ | **M1** | |
| | Through ( 2, –1 , 1) Hence $7x + y + 6z = 19$ | **A1** | |
| | | **2** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 7(i) | $\begin{pmatrix} 1 & -1 & -2 & 3 \\ 5 & -3 & -4 & 25 \\ 6 & -4 & -6 & 28 \\ 7 & -5 & -8 & 31 \end{pmatrix} \to \ldots \to \begin{pmatrix} 1 & -1 & -2 & 3 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ | **M1A1** | |
| | r(**A**) = 4 – 2 = 2 | **A1** | |
| | $x - y - 2z + 3t = 0$<br>$y + 3z + 5t = 0$ | **B1** | |
| | $z = \lambda,\ t = \mu \ \Rightarrow\ x = -\lambda - 8\mu,\ \ y = -3\lambda - 5\mu$ | **M1** | |
| | Basis for null space is $\left\{ \lambda \begin{pmatrix} -1 \\ -3 \\ 1 \\ 0 \end{pmatrix}, \mu \begin{pmatrix} -8 \\ -5 \\ 0 \\ 1 \end{pmatrix} \right\}$ $\begin{pmatrix} 19 \\ 0 \\ 5 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 19 \\ -8 \\ 1 \end{pmatrix}$ | **A1 A1** | OE |
| | | **7** | |
| 7(ii) | $\mathbf{A}\begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 21 \\ 24 \\ 27 \end{pmatrix}$ | **B1** | |
| | $\mathbf{x} = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -3 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -8 \\ -5 \\ 0 \\ 1 \end{pmatrix}$ | **M1A1FT** | OE |
| | | **3** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 8(i) | $I_2 = \int_0^{\frac{1}{4}\pi} \sec^2 x \mathrm{d}x = \left[\tan x\right]_0^{\frac{1}{4}\pi} = 1$ | **M1A1** | |
| | | **2** | |
| 8(ii) | $I_n = \int_0^{\frac{1}{4}\pi} \sec^{n-2} x . \sec^2 x \mathrm{d}x$ | **M1** | |
| | $= \left[\sec^{n-2} x \tan x\right]_0^{\frac{1}{4}\pi} - \int_0^{\frac{1}{4}\pi} (n-2) \sec^{n-3} x (\sec x \tan x) \tan x \mathrm{d}x$ | **M1A1** | |
| | $= \left[\sec^{n-2} x \tan x\right]_0^{\frac{1}{4}\pi} - (n-2) \int_0^{\frac{1}{4}\pi} \sec^{n-2} x \left(\sec^2 x - 1\right) \mathrm{d}x$ | **M1A1** | |
| | $\Rightarrow (n-1) I_n = 2^{\frac{1}{2}n-1} + (n-2) I_{n-2}$ | | AG |
| | | **5** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 8(iii) | Volume of revolution $=\pi\int y^2\mathrm{d}x=\pi\int_0^{\frac{1}{4}\pi}\sec^6 x\mathrm{d}x$ | **M1** | |
| | $3I_4=2+2\times1\Rightarrow I_4=\frac{4}{3}$ | **M1** | |
| | $5I_6=4+4\times\frac{4}{3}\Rightarrow I_6=\frac{28}{15}$ | **M1** | |
| | Volume of revolution $=\frac{28\pi}{15}$ | **A1** | |
| | | **4** | |

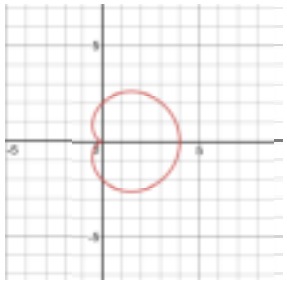
| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 9(i) | Degree of numerator < degree of denominator $\Rightarrow y = 0$ is horizontal asymptote. | **B1** | |
| | $(x+1)(x-2)=0 \Rightarrow x=-1$ and $\Rightarrow x=2$ are vertical asymptotes. | **B1** | |
| | | **2** | |
| 9(ii) | $yx^2-(y+3)x+9-2y=0$ | **M1** | |
| | No points on $C$ if $(y+3)^2-4y(9-2y)<0$ | **M1** | |
| | $\Rightarrow 9y^2-30y+9<0 \Rightarrow 3y^2-10y+3<0$ | **A1** | |
| | $\Rightarrow (3y-1)(y-3)<0 \Rightarrow \frac{1}{3}<y<3$ | **A1** | AG |
| | | **4** | |
| 9(iii) | $\frac{dy}{dx}=0 \Rightarrow 3(x^2-x-2)-(3x-9)(2x-1)=0$ | **B1** | |
| | $\Rightarrow \ldots \Rightarrow (x-1)(x-5)=0$ | **B1** | |
| | $\Rightarrow$ Turning points are (1,3) and $\left(5,\frac{1}{3}\right)$ . | **B1** | |
| | | **3** | |

| Question | Answer | | Marks | Guidance |
|---|---|---|---|---|
| 9(iv) |  | Axes, asymptotes and points on axes (0, 4.5) (3,0). | **B1** | |
| | | RH branch ; Other two branches | **B1B1** | |
| | | | **3** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 10(i) | $\sin 5\theta = Im(c+\mathrm{i}s)^5 =$ $Im\left(c^5+5c^4\mathrm{i}s+10c^3(\mathrm{i}s)^2+10c^2(\mathrm{i}s)^3+5c(\mathrm{i}s)^4+(\mathrm{i}s)^5\right)$ | **B1** | SOI |
| | $\sin 5\theta = 5c^4s-10c^2s^3+s^5$ | **M1A1** | |
| | $=s\left(5\left[1-s^2\right]^2-10s^2\left[1-s^2\right]+s^4\right)$ | **M1** | |
| | $=\ldots=5\sin\theta-20\sin^3\theta+16\sin^5\theta$ | **A1** | AG |
| | | **5** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 10(ii) | If $\theta = 0$ ,$\pm\frac{1}{5}\pi$ ,$\pm\frac{2}{5}\pi$ then $\sin 5\theta = 0$ | **B1** | |
| | $\Rightarrow 16s^5 - 20s^3 + 5s = 0$, where $s = \sin\theta$,<br>$\Rightarrow s\left(16s^4 - 20s^2 + 5\right) = 0$ | **B1** | |
| | $s = 0 \Rightarrow \theta = 0$ | **B1** | |
| | Hence roots of $16s^4 - 20s^3 + 5 = 0$ are $\pm\sin\frac{1}{5}\pi$ ,$\pm\sin\frac{2}{5}\pi$ | | AG |
| | | **3** | |
| 10(iii) | Since $\sin\frac{4}{5}\pi = -\sin\left(-\frac{1}{5}\pi\right)$ and $\sin\frac{3}{5}\pi = -\sin\left(-\frac{2}{5}\pi\right)$ | **B1** | |
| | $\sin\left(\frac{4}{5}\pi\right)\sin\left(\frac{3}{5}\pi\right)\sin\left(\frac{2}{5}\pi\right)\sin\left(\frac{1}{5}\pi\right) =$<br>$\sin\left(-\frac{1}{5}\pi\right)\sin\left(-\frac{2}{5}\pi\right)\sin\left(\frac{1}{5}\pi\right)\sin\left(\frac{2}{5}\pi\right) = \frac{5}{16}$ | **M1A1** | |
| | $\sin^2\frac{1}{5}\pi + \sin^2\frac{2}{5}\pi = -\frac{(-20)}{16} = \frac{5}{4}$ | **A1** | |
| | | **4** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 11E(i) | $\mathbf{Ae} = \lambda\mathbf{e}$ and $\mathbf{Be} = \mu\mathbf{e}$ | **M1A1** | |
| | $\mathbf{ABe} = \mathbf{A}\mu\mathbf{e} = \mu\mathbf{Ae} = \mu\lambda\mathbf{e} = \lambda\mu\mathbf{e}$ | **M1** | AG |
| | | **3** | |
| 11E(ii) | $(\lambda+1)(\lambda^2-5\lambda+6)=0$ | **A1** | |
| | $(\lambda+1)(\lambda-2)(\lambda-3)=0$ | **A1** | |
| | $\lambda=-1,2,3.$ | **M1** | |
| | Eigenvectors are $\begin{pmatrix}1\\-1\\0\end{pmatrix}$, $\begin{pmatrix}1\\-1\\1\end{pmatrix}$ and $\begin{pmatrix}1\\0\\1\end{pmatrix}$ respectively. | **A1A1** | Uses either vector product or equations to find eigenvectors |
| | | **6** | |
| 11E(iii) | $\begin{pmatrix}3&6&1\\1&-2&-1\\6&6&-2\end{pmatrix}\begin{pmatrix}1\\-1\\0\end{pmatrix}=\begin{pmatrix}-3\\3\\0\end{pmatrix}\Rightarrow \mu_1=-3$ | **M1** | |
| | Similarly, other two eigenvalues of **B** are – 2 and 4. | **A1** | |
| | Eigenvalues of **AB** are 3 , – 4 and 12 | **A1** | |
| | Corresponding eigenvectors are $\begin{pmatrix}1\\-1\\0\end{pmatrix}$, $\begin{pmatrix}1\\-1\\1\end{pmatrix}$ and $\begin{pmatrix}1\\0\\1\end{pmatrix}$. | **A1** | |
| | | **4** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 11OR |  | **B1** | Closed curve starting and ending at pole, in approximately correct location. |
| | | **B1** | Cardioid with indication of correct scale. |
| | | **2** | |
| 11OR(ii) | $r = a(1+\cos\theta) \Rightarrow \sqrt{x^2+y^2} = a\left(1+\frac{x}{\sqrt{x^2+y^2}}\right)$ | **M1** | |
| | $x^2+y^2 = a(x+\sqrt{(x^2+y^2)})$ | **A1** | Substitutes for r **and** cos(θ) |
| | | **2** | |

| Question | Answer | Marks | Guidance |
|---|---|---|---|
| 11OR(iii) | Sector area $=\frac{a^2}{2}\int_0^{\frac{1}{3}\pi}\left(1+2\cos\theta+\cos^2\theta\right)\mathrm{d}\theta$ <br> $=\frac{a^2}{2}\int_0^{\frac{1}{3}\pi}\left(\frac{3}{2}+2\cos\theta+\frac{\cos 2\theta}{2}\right)\mathrm{d}\theta$ | **M1A1** | |
| | $=\frac{a^2}{2}\left[\frac{3\theta}{2}+2\sin\theta+\frac{\sin 2\theta}{4}\right]_0^{\frac{1}{3}\pi}$ | **M1** | |
| | $=\frac{a^2}{16}\left(4\pi+9\sqrt{3}\right)$ | **A1** | |
| | | **4** | |
| 11OR(iv) | Arc length $=\int_0^{\frac{1}{3}\pi}\sqrt{a^2\left(1+2\cos\theta+\cos^2\theta\right)+a^2\left(-\sin\theta\right)^2}\,\mathrm{d}\theta$ | **M1A1** | |
| | $=a\int_0^{\frac{1}{3}\pi}\sqrt{2+2\cos\theta}\,\mathrm{d}\theta$ | **A1** | |
| | $=a\left[4\sin\frac{\theta}{2}\right]_0^{\frac{1}{3}\pi}=2a$ | **M1A1** | |
| | | **5** | |